TOWARD BUILDING A THEORY OF MATHEMATICAL MODELLING

Jennifer A. Czocher

Texas State University

This study utilized an innovative data analysis approach to examine how engineering undergraduates engaged in mathematical modelling. Individual modelling routes were constructed via modelling activity diagrams and were used to critically examine the theoretical framework. Implications for the theoretical model are offered along with implications for future research.

The purpose of this study is to contribute to the growing body of work indicating that the mathematical thinking which supports mathematical modelling is not regular and cyclic, but is instead idiosyncratic and context-dependent. Theories of how individuals engage in mathematical modelling – the practice of combining mathematical and nonmathematical knowledge to develop mathematical explanations for natural phenomena – assert regularities in the construction of mathematical models. In particular, claims have been made that the process is cyclic and iterative, involving a sequence of stages of model construction and mathematical activities that transform them (Blum & Leiß, 2007). Others suggest that this may not be the case (Ärelebäck, 2009; Borromeo-Ferri, 2007). As the next step in developing a model of individuals' mathematical modelling activity, the existing theory must be evaluated in light of a broader observational base and analytic techniques. Thus, the theory of model construction was adopted both as a theoretical framework to guide data collection and analysis and as a research framework.

This study is a close, systematic inspection of the mathematical thinking that constitutes the activities involved in mathematical modelling. Two questions guided task selection and data analysis: (i) Is mathematical modelling a regular, quasiperiodic process? (ii) How do individuals engage in mathematical modelling tasks?

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

This study uses the theory of model construction as a research framework and as a theoretical framework. Mathematical modelling has been theorized as an iterative, cyclic process that renders a real world problem as a mathematically well-posed problem that is then analysed mathematically and its solution interpreted in terms of real world constraints. The model is then validated against real-world observations and rejected or revised. Typically, models begin as crude representations or explanations and become more detailed and sophisticated after multiple iterations of this process. A schematic describing the process is given in Figure 1 (Blum & Leiß, 2007). The mathematical modelling cycle (MMC) is a series of six stages of model construction (stages [a] – [f]) sequentially linked by a series of six transitions (transitions [1] – [6]).

2014. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 2, pp. 353-360. Vancouver, Canada: PME.

Tables 1 and 2 give brief descriptions of each of the stages and transitions among them. The MMC was adopted as theoretical framework for this study.

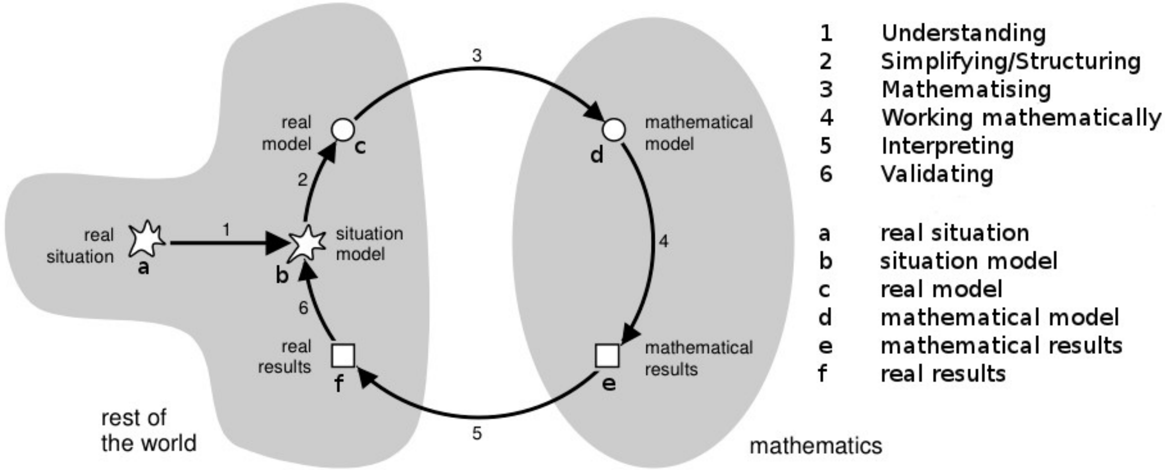

Figure 1: The mathematical modelling cycle (Blum & Leiß, 2007)

Stage of Model	Definition
[a] real situation	situation, as observed in the world
[b] situation model	conceptual model of problem
[c] real model	idealized version of the problem (serves as basis for mathematization)
[d] mathematical model	model in mathematical terms
[e] mathematical results	answer to mathematical problem
[f] real results	answer to real problem

Table 1: Stages of Model Construction

Transition	Captures	Sample Indicator
[1] understanding	forming an idea about what the problem is asking for	reading the task
[2] simplifying & structuring	identify critical components of the problem situation	making assumptions to "simplify" the problem
[3] mathematizing	represent the idealized real model mathematically	writing mathematical representations of ideas
[4] working mathematically	mathematical analysis	explicit algebraic or arithmetic manipulations
[5] interpreting	recontextualizing the mathematical result	speaking about results in context of the problem
[6] validating	verifying results against the real world	implicit or explicit statements about the reasonableness of the answer

Table 2: Transitions among stages in the modelling process

Using a similar theoretical framework, Borromeo-Ferri (2007) suggested individual modelling routes as a means for documenting individuals' cognition during mathematical modelling. An individual modelling route is "the individual modelling process on an internal and external level," (p. 265) though only *visible modelling routes* (verbal utterances or external representations) can be observed. Borromeo-Ferri's individual modelling routes took the form of arrow diagrams which traced the modeller's work through the modelling cycle. Findings suggested that modelling routes are idiosyncratic rather than cyclic as predicted by theory. Additionally, it was suggested that a change in representation might aid in understanding how individuals combined their mathematical and nonmathematical knowledge.

Ärelebäck (2009) adapted problem solving activity diagrams (Schoenfeld, 1985) to create Modeling Activity Diagrams (MADs) in order to study groups' modelling activity. MADs map a modelling event to a staff where each line is colour-coded to the modelling transition that leads the line. The result is a concise graphical representation of model construction with the advantage of providing chronological structure to the modelling activity.

The MADs track the length of time that the solver(s) were engaged in each activity. There are two drawbacks to this approach. First, the researcher cannot precisely determine when a particular transition begins or ends. Second, it is unclear how the time unit is meaningful because duration of the transition may not correspond to its meaningfulness mathematically or to its import to modelling progress. For example, if an individual spends a long time working mathematically, it may indicate a task with many steps to analysis; it may indicate an individual's difficulty in carrying out that analysis; or it could indicate that the individual paused to think about something else though outwardly he appeared to be on task. To further complicate matters, an individual may be engaged in more than one activity simultaneously or not visibly engaged in any activity. Both issues are important to consider because interpretations of the MADs are highly sensitive to the grain size of analysis and to whether verbal and written externalizations of the model are treated equivalently.

This study responds to Borromeo-Ferri's call for examining modelling routes and it uses MADs to do so. By reducing the grain size of the analytic unit and treating verbal and written externalizations of the mathematical model with equal weight, analysis of individual modelling routes and MADs can strengthen theoretical models of individuals' mathematical thinking during mathematical modelling.

METHODOLOGY

Participants were four engineering majors enrolled in a course on differential equations at a large US Midwestern university. A calculus screening test based on the Calculus Concept Inventory (Epstein, 2006) was administered to volunteers and four participants were selected such that two had high performance and two had low performance. The individuals were purposefully selected to maximize variation in

their backgrounds and ability levels. All participants were male: Mance (sophomore, environmental engineering, low performance), Trystane (sophomore, mechanical engineering, low performance), Orys (freshman, chemical engineering, high performance), Torrhen (freshman, electrical engineering, high performance).

Seven one-on-one, semi-structured, task-based interviews and one follow up member check interview were conducted. The goal of each primary interview was to elicit modelling activity. Interview techniques were drawn from experiment principles such as cross-fertilization and thought experiments (Brown, 1992). Nineteen tasks were designed to elicit the stages and transitions of the MMC and were developed through an iterative process starting with gathering modelling tasks from textbooks and research papers, mapping expected student responses against the MMC, and then review by a panel of mathematics educators and mathematicians. Many were solvable through multiple methods ranging from arithmetic to differential equations. Fourteen tasks were administered and 7 eliciting all transitions (some of the 14 focused on only one) were used for analysis.

Interviews were video recorded, transcribed, and reduced to MADs in the following way. The unit of analysis was one student working on one task, termed an event. The transcript of each event was parsed into a series of mathematically complete verbalized or written ideas. Using the method of constant comparison, a rubric of indicators for each transition activity in the MMC was developed and these indicators were applied to each unit. Sample indicators are given in Table 2.

The MADs were constructed in MATLAB as two dimensional graphs. Time (in seconds) is along the horizontal axis and transitions from the MMC along the vertical axis. Each transition was assigned a colour and vertical position. Each analytic unit was assigned the ordered pair (timestamp, transition). In this way, interview protocols were reduced to individual modelling routes represented as MADs (Figures 2 – 4). Each coloured mark represents when that particular transition between two stages of model building began. Elongated marks are artefacts of the scale do not indicate the length of time an individual was engaged in an activity. This serves to emphasize sequencing of transitions through the MMC, when the MADs are read left-to-right, rather than relative lengths of time spent executing each activity.

Each event was regarded as a product of some configuration of personal experiences, mathematical knowledge, and nonmathematical knowledge. These configurations were then examined for regularities across events and for divergences from predictions of the MMC. To accomplish the latter, an "ideal" MAD (Figure 2) was generated from the idealized MMC.

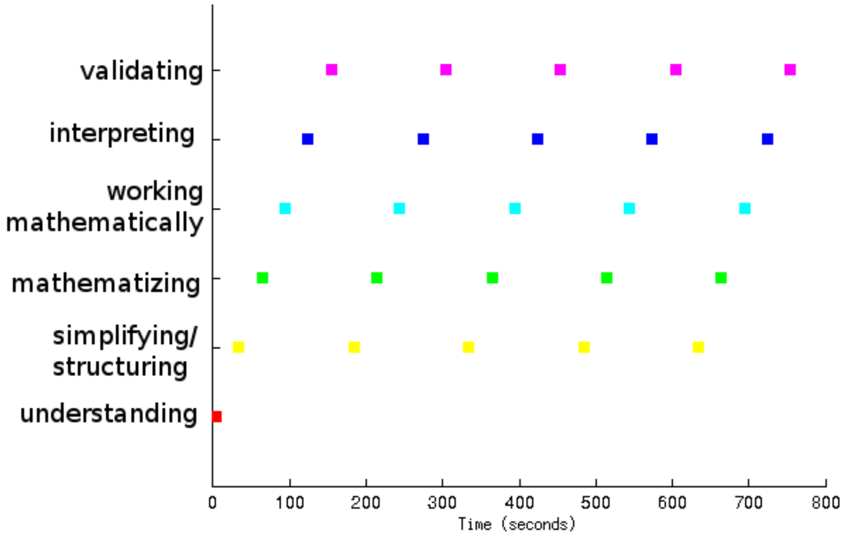

Figure 2: MAD corresponding to the MMC

RESULTS

This section presents the findings of a cross-event analysis of all MADs, but due to space constraints, the MADs for only one task, the Falling Body Problem (Figure 3 and 4), are displayed. This task was chosen because its MADs most clearly show each of the five deviations of the data from theoretical predictions. The task was: *On November 20, 2011, Willie Harris, 42, a man living on the west side of Austin, TX died from injuries sustained after jumping from a second floor window to escape a fire at his home. What was his impact speed?*

This was a standard dynamics problem (a critical variable is *time*) from physics and calculus solvable using kinematics, energy, or first-order differential equations. Mance used kinematics and made only one pass through the modelling cycle. In his MAD (Figure 3), each of the transitions fades in and out over time. That is, simplifying/structuring ceased as mathematizing took over and mathematizing faded out as working mathematically dominated. Torrhen, Trystane, and Orys made multiple passes through the modelling cycle as they changed their approaches by considering the effect of air resistance. Trystane refined his model multiple times, changing his conceptual model from *energy* to *kinematics* to *differential equations*, ultimately considering variables such as force-due-to-drag and surface area of the falling object. For all students except Mance, understanding, simplifying/structuring, and validating were exhibited frequently and consistently throughout the MADs.

The MADs provide an overview of an individual's modelling activity. The ideal MAD (Figure 2) exhibits a sawtooth pattern corresponding to the individual traversing the MMC over and over again as he adjusts the model to make it more accurate. Considering the MAD as encoding information about the individual's mathematical thinking during modelling, then this pattern is a signal and deviations from it are noise. Analysis revealed five deviations from theoretical prediction and possible reasons for the noise are discussed below.

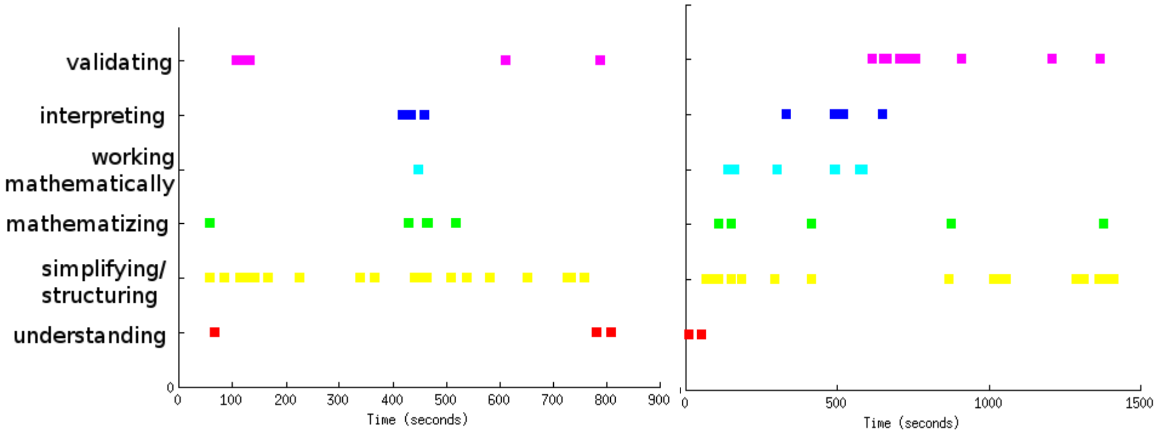

Figure 3: MADs for Mance (left) and Torrhen (right)

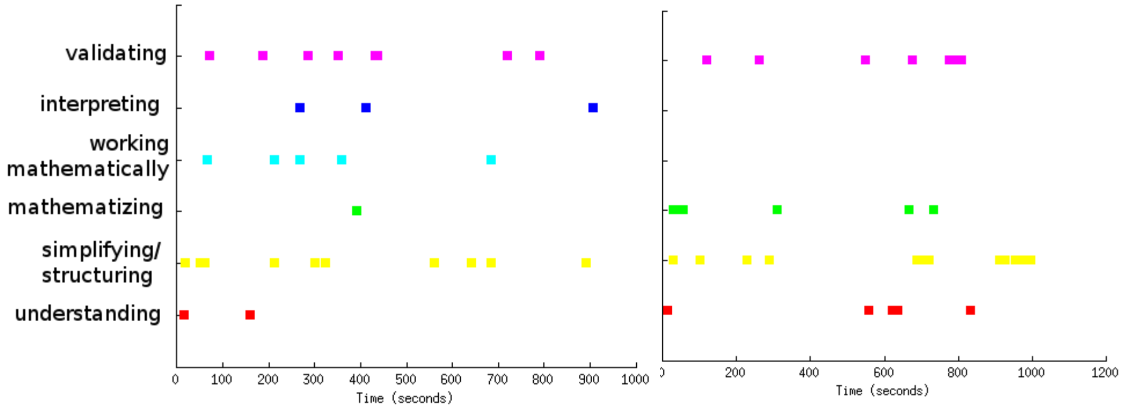

Figure 4: MADs for Orys (left) and Trystane (right)

First, the MADs show that an individual's movements are not solely "forward" in the modelling cycle. The individual may go back to consider previous stages of the model, may consider multiple stages simultaneously, or may skip transitions altogether. For example, at 600s, Torrhen was considering important variables and relationships while he is mathematizing them and at around 1000s, he rereads the problem statement (understanding) but returned directly to working mathematically without exhibiting the transitions in between.

Second, the sawtooth pattern is present, but noisy and spread out over time. For example, Mance's MAD progresses through the transitions in the MMC over the 900s, but is neither linear nor cyclic. Mance made corrections to his mathematical work, but did not revise his model. When revisions occur, they appear as bands of activity rather than neat, linear, sequential steps of a sawtooth pattern. The macroscopic banding structure is most clear in Trystane's MAD. Trystane's MAD exhibits three bands (0 – 300s, 400 – 900s, and 1000 – 1400s), but they are difficult to distinguish because understanding, simplifying/structuring, and validating occur throughout the MAD.

Third, understanding activity is present throughout the MADs; its appearance does not correspond to the start of a cycle. This is visible in Torrhen's, Orys's, and Trsytane's MADs. The most common source of this noise was the student returning to read the problem statement. Some instances could be considered monitoring because the

individual compared his goal or subgoal to the task. In other instances, the problem statement was used to find more information for the simplifying/structuring phase.

Fourth and fifth, there is increased presence of simplifying/structuring and validating activities. These features are evident in all of the MADs. Both transitions typically occurred throughout the MADs. In MADs presented here, with the exception of Mance, they occurred throughout each pass through the MMC. Taken together, this suggests that the individuals were consistently checking throughout modeling whether the variables and relationships assumed to be important in the model were necessary and sufficient. That is, it was not an activity that occurred only at the end of a cycle.

Validating often occurred at sites where there were no real results to verify. For example, Torrhen checked the accuracy of a computation at 200s prior to obtaining a result to evaluate in terms of the real world. At 100s, Orys engaged in validating activity immediately after reading the task when he questioned the legitimacy of the task itself asserting "most people would survive from jumping from a second floor window." The individuals were indeed validating other aspects of their models and how real world information might relate to their models. A focused investigation is necessary to determine the nature of the role of validating in mathematical modelling and in particular its relationship to simplifying/structuring.

DISCUSSION AND CONCLUSIONS

Analysis shows that the mathematical thinking involved in mathematical model construction is not sequential nor quasi periodic. The macroscopic structure of the MADs echo the idealized MMC. However the kind and quantity of deviations of the observed individual modelling routes from the model's predictions suggest that there are critical phenomena which are unaccounted for by the theoretical framework. These findings confirm prior conjectures that "the view presented on modelling as a cyclic process is highly idealised, artificial, and simplified" (Ärlebäck , 2009, p. 353). This is expected, since models are representations of simplified versions of reality. These discrepancies should lead to revision of the MMC.

There is tension between a desire for an accurate, predictive model and a model that is too complex or situation-specific to be of general use. Zbiek and Connor (2006) responded to the irregularities within students' work by introducing more stages and transitions which may collapse when an individual is facing a routine task. Collapsing would be consistent with the appearance of Mance's MAD for the Falling Body Problem. The MMC accurately describes the practice of modelling, but requires additional consideration to account for factors like individuals' prior knowledge, experiences modelling, and the purpose of the model.

The MADs and their subsequent analysis are a product of how the list of indicators operationalized the transitions in the MMC and grain-size of the unit of analysis. These modifications were necessary to capture the students' mathematical work and thinking, especially in the advanced mathematical settings not yet explored with the MMC. In

particular, working mathematically was defined broadly to include observations like *using deductive reasoning*; validating was redefined in terms of indicators instead of by the MMC. One avenue for future research is to use similar analytic techniques to examine the nature and role of validating activity and how it interacts with other mathematical activities. Another is to use the MADs to investigate where validating and simplifying/structuring occur in the modelling sequence as a means to examine how individuals combine mathematical and nonmathematical knowledge.

The goal of this line of research is to model individuals' mathematical thinking as they conduct mathematical modelling. The MMC provides an overview of its macroscopic structure. There is enough variation across tasks and individuals that we cannot claim that a cyclic, quasiperiodic description provides the only theoretical view of how individuals combine mathematical and nonmathematical knowledge. Mathematical modelling is a complex process and there is much work to be done to build a comprehensive theory of mathematical modelling.

References

Arleack, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Mathematics Enthusiast*, *6*(3), 331-364.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling: education, engineering and economics* (pp. 222-231). Chichester, UK: Horwood.

Borromeo-Ferri, R. (2007). Modelling problems from a cognitive perspective. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling: education, engineering, and economics* (pp. 260-270). Chichester, UK: Horwood.

Brown, A. L. (1992). Design experiments: theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences*, *2*(2), 141-178.

Epstein, J. (2006). The calculus concept inventory. In D. Deeds & B. Callen, *Proceedings of the National STEM Assessment Conference* (pp. 60-67). Washington, DC.

Schoenfeld, A. H. (1985). Metacognitive and epistemological issues in mathematical understanding. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 361-379). Hillsdale, NJ: Lawrence Erlbaum Associates.

Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modelling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics*, *63*(1), 89-112.